A note on some inequalities for the Tutte polynomial of a matroid

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\textbf{Abstract}

We prove that if a matroid $M$ contains two disjoint bases (or, dually, if its ground set is the union of two bases), then $T_M(a,a) \leq \max\{T_M(2a,0), T_M(0,2a)\}$ for $a \geq 2$. This resembles the conjecture that appears in C. Merino and D.J.A. Welsh, Forests, colourings and acyclic orientations of the square lattice, \textit{Annals of Combinatorics} \textbf{3} (1999) pp. 417–429: If $G$ is a 2-connected graph with no loops, then $T_G(1,1) \leq \max\{T_G(2,0), T(0,2)\}$. We conjecture that $T_M(1,1) \leq \max\{T_M(2,0), T_M(0,2)\}$ for matroids which contains two disjoint bases or its ground set is the union of two bases. We also prove the latter for some families of graphs and matroids.

\textbf{Keywords:} Matroid, Tutte polynomial

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1 Introduction

The Tutte polynomial is a two variable polynomial which can be defined for a graph $G$ or, more general, a matroid $M$. The most relevant feature of the Tutte polynomial is that when evaluated on different points and along several algebraic curves we obtained interesting combinatorial information about the graph or matroid. For instance, the Tutte polynomial of a graph $G$ along the line $y = 0$ is, essentially, the chromatic polynomial and along the line $x = 0$ is the flow polynomial. Also, important invariants in other fields can be presented as evaluations of the Tutte polynomial, for example the all terminal reliability of a network, the partition function of the $Q$-state Potts model and the weight enumerator of liner codes over $\text{GF}(q)$.

Thus, the Tutte polynomial has received a fair amount of attention by trying to get new interpretations or by understanding the structure of the Tutte polynomial. Here we concentrate on understanding the structure of the Tutte polynomial, more specifically the graph of the polynomial, that is the surface $\{T(x, y)\}_{(x,y) \in \mathbb{R}^2}$. Our results are still partial, but they seem to point to a theorem about the convexity of $\{T(x, y)\}_{(x,y) \in \mathbb{R}^2}$ in the positive quadrant.

2 Preliminaries

The Tutte polynomial is a matroid invariant over the ring $\mathbb{Z}[x, y]$. One of the simplest definitions, which is often the easiest way to prove properties of the Tutte polynomial, uses the notion of rank.

If $M = (E, r)$ is a matroid, where $r$ is the rank-function of $M$, and $A \subseteq E$, we denote $r(E) - r(A)$ by $z(A)$ and $|A| - r(A)$ by $n(A)$ (the function $n(A)$ is called the nullity of $A$).

Definition 2.1 The Tutte polynomial of $M$, $T_M(x, y)$, has the following expansion

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{z(A)} (y - 1)^{n(A)}.$$  

We require another (equivalent) definition of the Tutte polynomial but first we introduce the relevant notions.

Let us take a fixed ordering $\prec$ on the elements of $M$, say $E = \{e_1, \ldots, e_m\}$, where $e_i \prec e_j$ if $i < j$. Given a fixed basis $S$, an element $e$ is called internally active if $e \in S$ and it is the smallest edge in the only cocircuit disjoint from $S \setminus \{e\}$. Dually, an element $f$ is externally active if $f \notin S$ and it is the smallest element in the only circuit contained in $S \cup \{f\}$. We define $t_{ij}$ to
be the number of bases with \( i \) internally active elements and \( j \) externally active elements. In [7] Tutte defined \( T_M \) using these concepts. A proof of the equivalence with Definition 2.1 can be found in [1].

**Definition 2.2** If \( M = (E, r) \) is a matroid with a total order on its ground set, then

\[
T_M(x, y) = \sum_{ij} t_{ij} x^i y^j.
\]  

(2)

In particular, the terms \( t_{ij} \) are independent of the total order used on the ground set.

There are a number of identities that hold for the coefficients \( t_{ij} \). For a complete characterization of all the affine linear relations that hold among the coefficients \( t_{ij} \) see Theorem 6.2.13 in [3]. From there we extract the following relations.

**Theorem 2.3** If a rank-\( r \) matroid \( M \) with \( m \) elements has neither loops nor isthmuses, then

(i) \( t_{ij} = 0 \), whenever \( i > r \) or \( j > m - r \);
(ii) \( t_{r0} = 1 \) and \( t_{0,m-r} = 1 \);
(iii) \( t_{rj} = 0 \) for all \( j > 0 \) and \( t_{i,m-r} = 0 \) for all \( i > 0 \).

Further details of many of the concepts treated here can be found in [8] and [3]. For matroid theory we suggest [6].

### 3 Some inequalities for the Tutte polynomial

From the result in the previous section it is easy to prove the following:

**Theorem 3.1** If a matroid \( M \) has neither loops nor isthmuses, then

\[
\max\{T_M(4, 0), T_M(0, 4)\} \geq T_M(2, 2).
\]

Note that, for a matroid \( M = (E, r) \) with dual \( M^* = (E, r^*) \), the following inequalities are equivalent for any \( A \subseteq E \).

\[
|A| \leq |E| - 2(r(E) - r(A)), \quad (3)
\]
\[
|E \setminus A| \leq 2r^*(E \setminus A) \text{ and } (4)
\]
\[
z(A) + n(A) \leq m - r. \quad (5)
\]

We now restrict our attention to matroids \( M \) in which all subsets \( A \) of the ground set \( E \) satisfy the (equivalent) inequalities above. By a classical result
of J. Edmonds [4], these are the matroids that contain two disjoint bases; by duality, these are the matroids $M$ whose ground set is the union of two bases of $M^*$.

**Theorem 3.2** If a matroid $M$ contains two disjoint bases, then $t_{ij} = 0$, for all $i$ and $j$ such that $i + j > m - r$. Dually, if its ground set is the union of two bases, then $t_{ij} = 0$, for all $i$ and $j$ such that $i + j > r$.

Our main result is the following:

**Theorem 3.3** If a matroid $M$ contains two disjoint bases, then

$$T_M(0, 2a) \geq T_M(a, a),$$

(6)

for all $a \geq 2$. Dually, if its ground set is the union of two bases of $M^*$, then

$$T_M(2a, 0) \geq T_M(a, a),$$

(7)

for all $a \geq 2$.

As a corollary we get that for a matroid $M$, we have that $\max\{T_M(2a, 0), T_M(0, 2a)\} \geq T_M(a, a)$, for all $a \geq 2$ whenever $M$ is one of the following: an identically self-dual matroid $M$, a paving matroid, the uniform matroid $U_{r,n}$ for $0 \leq r \leq n$, or a rank-$r$ projective geometry over $GF(q)$ or its dual, for $r \geq 2$.

A similar result is true for a graphic matroid $M(G)$, $G$ a loopless, bridgeless graph, whenever $G$ is one of the following: a 4-edge-connected graph, a 2-connected chordal graph, a complete bipartite graph, a series-parallel graph, a cubic graph, a bipartite planar, a Laman graph, a triangulation, the wheel graph $W_n$, for $n \geq 2$, or the square lattice $L_n$, for $n \geq 2$.

4 A conjecture

After Theorem 3.3, it is natural to ask if inequalities (6) and (7) are valid for $a = 1$. For graphic matroids this is almost the conjecture made in [5].

**Conjecture 4.1** Let $G$ be a 2-connected graph with no loops, then

$$\max\{T_G(2, 0), T_G(0, 2)\} \geq T_G(1, 1).$$

(8)

Thus, Theorem 3.3 suggests that a more suitable conjecture for graphs and matroids is the following:

**Conjecture 4.2** If a cosimple matroid $M$ contains two disjoint bases then

$$T_M(0, 2) \geq T_M(1, 1).$$

(9)
Dually, if the ground set of a simple $M$ is the union of two bases, then

$$T_M(2, 0) \geq T_M(1, 1) .$$

We can prove that for a matroid $M$, we have that $\max\{T_M(2, 0), T_M(0, 2)\} \geq T_M(1, 1)$, whenever $M$ is one of the following: the uniform matroid $U_{r,n}$ for $0 \leq r \leq n$, the $n$-whirl $W^n$ or a Catalan matroid see [2]. A similar result is true for a loopless, bridgeless graph $G$, whenever $G$ is one of the following: the wheel graph $W_n$, a 3-regular graph with girth at least five, the complete graph $K_n$ and the complete bipartite graph $K_{n,m}$, for all $m \geq n \geq 2$. Along the way we prove some results that relates different constructions with Conjecture 4.2, for example the following:

**Theorem 4.3** If a matroid $M$ contains two disjoint bases, then its free extension $M + e$ satisfies Conjecture 4.2. Dually, if its ground set is the union of two bases, then the free coextension satisfies Conjecture 4.2.

**References**


